Universal behavior of QCD amplitudes at high energy from general tools of statistical physics

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Abstract

We show that high energy scattering is a statistical process essentially similar to reaction-diffusion in a system made of a finite number of particles. The Balitsky-JIMWLK equations correspond to the time evolution law for the particle density. The squared strong coupling constant plays the role of the minimum particle density. Discreteness is related to the finite number of partons one may observe in a given event and has a sizeable effect on physical observables. Using general tools developed recently in statistical physics, we derive the universal terms in the rapidity dependence of the saturation scale and the scaling form of the amplitude, which come as the leading terms in a large rapidity and small coupling expansion.

1 Introduction

Much progress has been made recently in understanding high energy hard scattering in QCD at or near the unitarity limit. General equations have been given

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by Balitsky [1] and by Jalilian-Marian, Iancu, McLerran, Leonidov, Kovner and Weigert (JIMWLK) [2,3,4] which generalize the Balitsky-Fadin-Kuraev-Lipatov (BFKL) evolution [5] to the region where unitarity (saturation) effects become important. The Balitsky-JIMWLK equations are nonlinear operator equations, while a "mean field" version of the equations, the Balitsky-Kovchegov (BK) [1,6] equation, is a nonlinear equation for the scattering amplitude and has been widely studied recently. The scattering amplitude which emerges from the BK equation is, in general terms, characterized by the energy (rapidity) dependence of the saturation momentum $Q_s(Y)$ [7,8,9,10,11,12], and by geometric scaling [13,9,10], the statement that the scattering amplitude $A(Q^2, Y)$ is equal to a function of a single variable $A(Q^2/Q_s^2(Y))$.

However, it has not been clear to what level the general properties of solutions to the BK equation are shared by solutions to the Balitsky-JIMWLK equations. This is the problem we address in this note. Our object is to describe the energy-dependence of the saturation momentum and the scaling properties of the scattering amplitude which should emerge from the Balitsky-JIMWLK equations. When viewed in a particular way the problem here looks identical to a class of problems studied recently in statistical physics [14,15]. In the statistical physics problems the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [16] approximately describes the time evolution of certain quantities in some discrete systems. In the QCD problem of dipole-dipole scattering the BK equation describes the rapidity and Q^2 dependence of the scattering amplitude. As has been noticed recently, the BK equation is in the same universality class as the FKPP equation and this fact gives a powerful and general derivation of the energy dependence of $Q_s(Y)$ and of geometric scaling [12].

The FKPP equation has limitations in applications to average quantities in discrete statistical systems. When the discreteness is not important, that is in a region where many "objects" are present, the FKPP equation is a good approximation to the actual evolution of the system. However, when only a few objects are involved discreteness effects are significant and the FKPP description breaks down [14,17]. A similar effect occurs in QCD evolution, and this is seen most easily by viewing the scattering of an elementary dipole of size r on an evolved dipole of initial size r_0 in terms of the $Y-$evolution of particular configurations of dipoles starting from r_0. (The scattering amplitude is then given by an average over all possible configurations.) The dipoles making up a configuration are the discrete elements of our system. The importance of fluctuations due to discretness in QCD evolution was first noticed by Salam from Monte-Carlo studies [18]. More recently, the importance of fluctuations has been reiterated in the context of non-linear evolution in Ref. [19], where the role of rare fluctuations in the approach of the S-matrix towards the unitarity limit has been discussed, and also in Ref. [4,20], where JIMWLK evolution has been reformulated as a random walk in some functional space, thus em-

phasizing its stochastic nature. This last formulation lies also at the basis of the numerical studies of JIMWLK evolution in Ref. [21].

So long as the dipole occupancy in a configuration is large compared to one, use of the BK equation should be a good approximation for the evolution of our configuration. However, when there are only a few dipoles of size r in our particular configuration BK evolution cannot be expected to be accurate. Indeed in a discrete picture occupancy can go below one only by becoming zero which stops the evolution along that path. Thus, using the BK equation with a cutoff when dipole occupancy is near one, or equivalently when the scattering amplitude for a particular configuration becomes of size α_s^2, exactly the same procedure used for discrete statistical systems [14,17], should be a good representation of the evolution of the system. This cutoff is essentially the same as that introduced in Ref. [22], and the present discussion can be viewed as a justification of the procedure used there at least for the calculation of the energy dependence of $Q_s(Y)$.

To also compute the dependence of the scattering amplitude upon the dipole size, and thus compare with the geometrical scaling form of the solution to BK equation, one needs to understand the fluctuations of the saturation momentum from one configuration to another. Here our control is less complete, and we rely on a scaling law recently seen in numerical simulations. The scale which emerges is equal to the square root of the value found in Ref. [22] where fluctuations at the boundary were not included.

Finally, it should be emphasized that our description is for a scattering at a definite impact parameter. A more complete discussion, including impact parameter dependences, will be given later [23].

2 High energy scattering as a statistical process

We consider the scattering of a dipole of variable size r (the probe) off a dipole of size r_0 (the target). A natural variable that will be used throughout is $\rho = \ln(r_0^2/r^2)$. We go to the rest frame of the probe so that the target carries all the available rapidity Y. The impact parameter b between the dipoles is fixed.

The target interacts through its quantum fluctuations, which at high energy are dominated by gluons. It proves useful to represent this set of partons by color dipoles [24]. This is possible in the dilute regime in which saturation effects (i.e. interactions among gluons inside the target wavefunction) are negligible, but the effects of fluctuations should on the contrary be important. Then, the dipole picture emerges in the large–N_c limit, in which gluons are

similar to zero–size $q\bar{q}$ pairs and non-planar diagrams are suppressed. The dipole approximation breaks down when the amplitudes approach their unitarity limits (indeed, in the considered frame, the unitarity corrections are tantamount to saturation effects in the target). However, that does not hamper getting the right asymptotics for physical quantities like the saturation scale since, as we shall see, this is controlled by the dynamics in the tail of the distribution at high transverse momenta, or small dipole sizes.

We denote by $T(r, r_0)$ the scattering amplitude of the probe off a given partonic realization $|\omega\rangle$ of the target (the dependence on b is understood). It is a random variable, whose probability distribution is related to the distribution of the different Fock state realizations of the target. The values of $T(r)$ range between 0 (weak interaction) and 1 (unitarity limit). $T(r)$ will be an essential intermediate quantity in our calculations, but it is not an observable. The physical dipole-dipole scattering amplitude $A(r, Y)$ is the statistical average over all partonic fluctuations of the target at rapidity Y, i.e. $A(r, Y) = \langle T(r) \rangle_Y$.

When T is small, $T(r, r_0) = \sum T_{el}(r, r_i)$, where i labels the dipoles in the Fock state of the target at the time of the interaction. T_{el} is the elementary dipole interaction and is essentially local in impact parameter. T_{el} behaves like

$$T_{el}(r, r_i) \sim \alpha_s^2 \frac{r_<^2}{r_>^2} \ , \ \text{ with } r_< = \min(|r|, |r_i|) \ , \ r_> = \max(|r|, |r_i|) \qquad (1)$$

when the dipoles overlap, and vanishes otherwise (see the insert in Fig. 1). We have neglected $\mathcal{O}(1)$ factors and logarithms, but these approximations do not affect the results that we shall obtain, which are largely independent of the details. Eq. (1) shows that the amplitude $T(r, r_0)$ is simply counting the number $n(r, r_0)$ of dipoles of size r within a disk of radius r centered at the impact parameter of the external dipole (the dipole occupation number):

$$T(r, r_0) \sim \alpha_s^2 \, n(r, r_0) \ . \qquad (2)$$

Note that $n(r, r_0)$ can take only discrete values. Thus, in this description, fluctuations in T emerge naturally as fluctuations in the *particle* (here, dipole) *number*, which should be especially important in the regime where $n \sim \mathcal{O}(1)$. The unitarity bound on T implies that $n(r, r_0)$ is also constrained by an upper bound $N \sim \mathcal{O}(1/\alpha_s^2)$.

The dipole picture also provides us with the evolution law for the dipole distribution with increasing rapidity. Consider a small increment dY of the total rapidity from a boost of the target. Then, each of the dipoles r already present in the wave function from the previous evolution and for which $n(r, r_0) \ll 1/\alpha_s^2$ may split into two new dipoles, of respectives sizes z and $r - z$, with a differ-

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ential probability[24]

$$dP = dY \frac{\bar{\alpha}}{2\pi} \frac{r^2}{z^2(r-z)^2} d^2z = [\lambda\,\bar{\alpha}dY] \times [p(z, r-z|r)\,d^2z] \qquad (3)$$

where $\bar{\alpha} = \alpha_s N_c/\pi$. We see that $\bar{\alpha}Y$ is the natural evolution variable: we will call it "time". The second equality in Eq. (3) expresses dP as the product of the inclusive probability of splitting[3]

$$\lambda = \int \frac{d^2z}{2\pi} \frac{r^2}{z^2(r-z)^2} \qquad (4)$$

in the time interval $\bar{\alpha}\,dY$, by the conditional distribution of the sizes of the produced dipoles

$$p(z, r-z|r)\,d^2z = \lambda^{-1} \frac{r^2}{z^2(r-z)^2} \frac{d^2z}{2\pi} \ . \qquad (5)$$

By applying Eqs. (3)–(5) to any of the dipoles r_i present in the wavefunction, one can easily deduce the evolution law for the dipole configuration as a whole. This leads to a description of the (dilute tail of the) target wavefunction as a stochastic ensemble of dipole configurations endowed with a probability distribution which evolves with Y according to a master equation [25]. This picture, which is similar to certain problems in statistical physics, is particularly appropriate for a study of fluctuations, since the discreteness of the dipole number is explicit. However, this picture breaks down, as anticipated, when the dipole occupation numbers — which in the dilute regime rise exponentially with Y — become of $\mathcal{O}(1/\alpha_s^2)$, and saturation effects start to play a role. But in this high density regime, one can rely on a different formalism, the color glass condensate [3], which is an effective theory for gluon correlations in the target wavefunction at small x, and is endowed with a *functional* evolution equation — the JIMWLK equation [2,3,4] — which shows how these correlations change under a boost. When this formalism is applied to the scattering between the color glass (the target) and a set of dipoles (the projectile), the saturation effects encoded in the JIMWLK equation are converted into unitarity effects in the evolution of the scattering amplitudes. The latter are thus found to obey an infinite hierarchy of evolution equations originally derived by Balitsky [1].

In what follows, we shall need only the first equation in this hierarchy, which applies when the projectile is a single dipole. This equation is most easily obtained by using the rapidity increment dY to accelerate the projectile (the dipole of size r). Within the rapidity interval dY, either the dipole does not split, in which case its scattering amplitude $T(r) \equiv T(r, r_0)$ remains unchanged, or it splits, in which case $T(r)$ is replaced by the scattering amplitude

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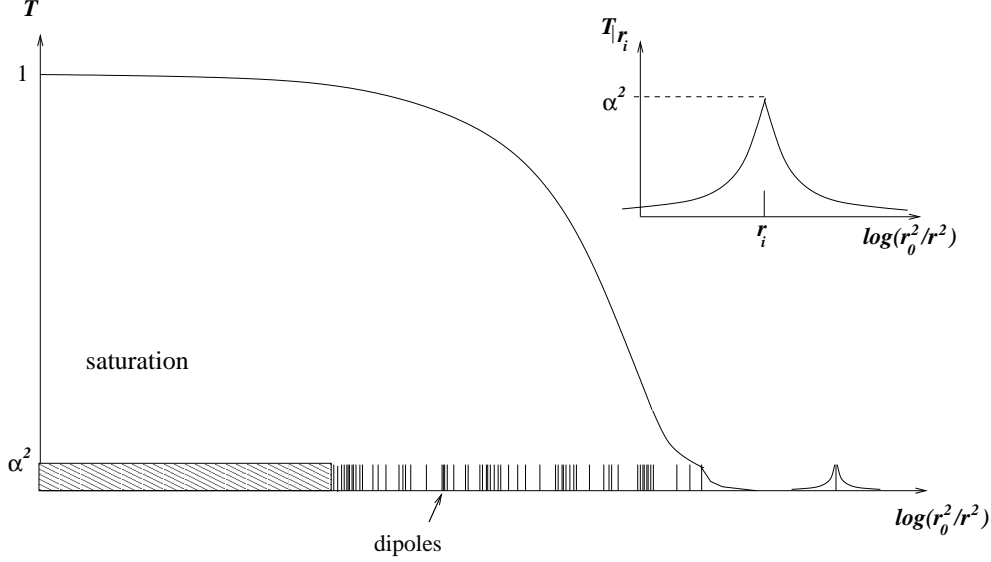[3] An ultraviolet cutoff is understood in Eq. (4). It disappears in physical quantities.

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Fig. 1. The amplitude T for a typical partonic realization as a function of $\rho = \ln(r_0^2/r^2)$. The individual dipoles seen at impact parameter b are represented by a short vertical line. The straight line is the sum of their contributions to the amplitude. In the saturation regime, the dipole description breaks down, that is indicated by the filled box. *Upper right corner:* the contribution of a single dipole to T, Eq. (1).

of the two child dipoles. This leads to the following evolution law

$$T(r)|_{Y+dY} = \begin{cases} T(r)|_Y & \text{with probability } 1 - \lambda\,\bar{\alpha}dY \\ T(z) + T(r-z) - T(z)T(r-z)|_Y & \text{with probability } \lambda\,\bar{\alpha}dY \end{cases}$$
(6)

where z is distributed according to $p(z, r-z|r)\,d^2z$. Taking the limit $dY \to 0$ and replacing λ and p from Eqs.(4),(5), one gets [4]

$$\partial_Y \langle T(r)\rangle_Y = \frac{\bar{\alpha}}{2\pi} \int d^2z \frac{r^2}{z^2(r-z)^2} \Big(\langle T(z)\rangle_Y + \langle T(r-z)\rangle_Y - \langle T(r)\rangle_Y$$
$$- \langle T(z)T(r-z)\rangle_Y \Big). \quad (7)$$

As anticipated, Eq. (7) is not a closed equation for $\langle T\rangle$: it depends upon the correlator $\langle T(z)T(r-z)\rangle_Y$. A mean field approximation $\langle T(z)T(r-z)\rangle \simeq \langle T(z)\rangle\langle T(r-z)\rangle$ would cast Eq. (7) into a closed form, known as the Balitsky-Kovchegov (BK) equation [1,6]. The linearized form of Eq. (7) is recognized as the (dipole version of) BFKL equation [5].

Let us finally discuss the typical shape of $T(r)$ as resulting from the previous considerations. It is a well known characteristic of the BFKL evolution that

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[4] The impact parameter dependence could be easily put back in Eq. (7). We have omitted it for simplicity and since it is enough for our purpose to assume locality of the evolution.

the most probable splittings are those in which both child dipoles have a size comparable to the size of the parent dipole (this can be also checked on Eq. (7)). Thus, if one starts with one dipole r_0 at $Y = 0$, then the main mechanism for the rise of $T(r)$ with Y is a growing diffusion around the size of the initial dipole r_0. Consequently, in a typical partonic configuration as obtained after a sufficiently large rapidity evolution, the dipoles appear to be densely distributed around the size r_0 (where $T(r)$ is large), but they become more rare with decreasing r (or increasing ρ), and for sufficiently large ρ one meets only rare fluctuations which involve one (or few) dipoles and for which $T(r) \simeq \alpha_s^2$. The typical partonic realization is shown in Fig. 1: it is a front which with increasing Y progresses towards larger values of ρ. We define the saturation scale $Q_s(Y)$ of a given partonic configuration by the position of this front, that is, by the value of the inverse dipole size for which T reaches some predefined number T_0 of order one: $T(1/Q_s(Y)) = T_0$. We also define $\rho_s(Y) = \log(r_0^2 Q_s^2(Y))$.

3   The energy dependence of the saturation scale

When $T(r)$ is of order α_s^2, the number of dipoles participating in the scattering is small, and fluctuations dominate the dynamics of $T(r)$. By contrast, when $T(r) \gg \alpha_s^2$, the fluctuations δT in T become relatively unimportant (since typically $\delta T \sim \alpha_s \sqrt{T}$), so the dynamics is self–averaging (for each individual front realization), and the mean field description of a given event becomes justified. Consequently, the evolution of T in the bulk of the front $T(r) \gg \alpha_s^2$ is essentially given by a mean field equation (the BK equation). It turns out that, for the purpose of computing the asymptotic energy dependence of the saturation momentum, one can still rely on a *modified* mean field approximation, which is obtained by introducing a factor $\Theta(T - \alpha_s^2)$ in the BK equation. The latter implements the fact that, in a real event, in which the occupation number is discrete, T cannot become less than α_s^2 (cf. the discussion in Sec. 2).

To appreciate the dynamical role of this cutoff, it is useful to notice an essential difference between the tail of a real event, and that of the solution to the BK equation: whereas the front generated by the BK equation has an exponential tail which extends up to arbitrarily large ρ (see Eq. (8) below), a real event, on the other hand, is like a histogram whose front is necessarily compact: for any Y, there exists a *foremost occupied bin* (f.o.b.) $\rho_{\text{f.o.b.}} \equiv \rho_{\text{f.o.b.}}(Y)$ such that $T(\rho_{\text{f.o.b.}})|_Y > 0$ and $T(\rho)|_Y = 0$ for any $\rho > \rho_{\text{f.o.b.}}$. This implies that the mechanism for front propagation is different in the two cases. For the mean field approximation, the dominant mechanism is the *local growth* within the tail of the distribution: at any $\rho \gg \rho_s(Y)$, the local amplitude rises very fast due to the BFKL instability, thus "pulling" the front towards the right. By contrast, in a real event, the local growth is not possible in the empty bins on

the right of the f.o.b., so the only way for the front to progress into those bins is via *diffusion*, i.e. via radiation from the occupied bins at $\rho < \rho_{\text{f.o.b.}}$. But since diffusion is less effective than the local growth, we expect the "velocity" $d\rho_s(Y)/dY$ of the front (i.e., the exponential growth rate for the saturation momentum) to be reduced in the real event as compared to the corresponding prediction of the BK equation.

A simple way to try and capture this physical situation in mathematical terms is to insert a cutoff $\Theta(T - \alpha_s^2)$ on the growth term in the BK equation, while allowing the diffusion there to remain operative even at arbitrarily small T. (This is possible after separating the local growth term from the diffusion term in the BFKL kernel with the help of a "diffusion approximation"; see, e.g., [12] for details.) This simple recipe was in fact invented by Brunet and Derrida [14] in the context of statistical physics. They studied the propagation of fronts in the presence of fluctuations associated with discreteness in a variety of physical situations (see Ref. [17] for a review). Using the mean field approximation to their dynamical equations together with a suitable cutoff, they were able to compute analytically the time evolution of the position of the front and of its bulk shape. Although there is so far no full mathematical justification of this procedure, it has been checked (through numerical calculations) that it yields indeed the right value for the velocity of the front at large times and for a large number of particles. This result is likely to be valid for all models that fall in the universality class of the stochastic FKPP equation [17].

We expect that, also in the present QCD context, that recipe give the right asymptotics of $\rho_s(Y) = \ln(r_0^2 Q_s^2(Y))$ for $\ln(1/\alpha_s^2) \gg 1$. Indeed, when viewed in the way exposed in Sec. 2, the rapidity evolution of T is essentially the same as the time evolution of say the particle number density of a system made of a number $N \sim 1/\alpha_s^2$ of diffusing and interacting particles, and hence belongs to the class of models studied by Brunet and Derrida.

We now turn to the practical computation of the rapidity dependence of the saturation scale. Since it amounts to solving the BK equation supplemented by a cutoff, it is useful to recall first the asymptotic solutions to the BK equation without a cutoff and the way how they set in (see e.g. [12] for details). As the BK equation falls into the universality class of the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [16], its asymptotic solution is a traveling wave, i.e. a uniformly translating front whose "position" is characterized by the logarithm of the saturation scale $\rho_s(Y)$. It moves with the "velocity" $d\rho_s(Y)/dY = \bar{\alpha}\chi(\gamma_0)/\gamma_0$ towards smaller dipole sizes. Here, $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$ is the ρ–moment of the dipole splitting probability (3), or, equivalently, the characteristic function of the BFKL kernel, and $\gamma_0 = 0.6275...$ solves $\chi'(\gamma_0) = \chi(\gamma_0)/\gamma_0$ [7]. The front exhibits a universal tail:

$$T(\rho, Y) \sim e^{-\gamma_0(\rho - \rho_s(Y))} \quad \text{for} \quad \rho - \rho_s \gg 1 \,. \tag{8}$$

These asymptotics set in diffusively and spread over the range $\rho - \rho_s(Y)$ within the time interval

$$\bar{\alpha}\Delta Y \sim \frac{(\rho - \rho_s)^2}{2\chi''(\gamma_0)} \ . \tag{9}$$

This process induces corrections to the asymptotic Y-dependence of the saturation scale of the form

$$\frac{d\rho_s(Y)}{dY} = \bar{\alpha}\frac{\chi(\gamma_0)}{\gamma_0} - \frac{3}{2\gamma_0}\frac{1}{Y} \ . \tag{10}$$

Note that the velocity and the shape of the front for $\rho \gg \rho_s$ are completely determined by the linearized (BFKL) equation, and do not depend on the exact form of the nonlinearities: this is a very important consequence of the nature of the propagation of the front, which is pulled along by its tail. It implies that for a number of physical quantities, such as ρ_s, we do not need to know the precise nonlinear mechanism that enforces unitarity of T. This means in particular, that the dipole picture is good enough for our purpose, although it is incomplete.

Equation (10) shows that the velocity of the front increases with Y up to its asymptotic value, which corresponds to the velocity of a front which has the shape (8) all the way down to $\rho \to \infty$. However, coming back to the physical situation where, due to the discreteness of the dipole number, the real front is a histogram, we see that T can assume the shape (8) only down to $T \sim \alpha_s^2$. Starting from the initial condition at $Y = 0$ and evolving it up to rapidity Y, the amplitude first grows until it reaches the unitarity limit $T = 1$ around $r \sim r_0$. Then the traveling wave front forms, and spreads from the point $\rho_s(Y)$ where $T \sim 1$ down to the point ρ at which $T \sim \alpha_s^2$. From Eq. (9) and from the shape of the asymptotic front Eq. (8), the latter process occurs within the rapidity interval

$$\bar{\alpha}\Delta Y = c\frac{\left[\ln(1/\alpha_s^2)/\gamma_0\right]^2}{2\chi''(\gamma_0)} \tag{11}$$

(c is a number of order one) during which the velocity of the front keeps increasing according to Eq. (10). But once the point where $T \sim \alpha_s^2$ is reached, the front cannot extend to even larger values of ρ (corresponding to lower values of T), at variance with the pure mean field case implemented by the BK equation. Accordingly, the front velocity cannot increase anymore. Thus the asymptotic Y-dependence of the saturation scale is

$$\frac{d\rho_s(Y)}{dY} = \bar{\alpha}\frac{\chi(\gamma_0)}{\gamma_0} - 3c\,\bar{\alpha}\frac{\gamma_0\chi''(\gamma_0)}{\ln^2(1/\alpha_s^2)} \ . \tag{12}$$

The calculation of c requires a proper account of the exact shape of the front, and yields $c = \pi^2/6$ [14,22].

The result (12) is identical to the one obtained in the mean field approach of

Ref. [22], where the effects of discreteness have been simulated by introducing a second barrier on the phase–space for BFKL evolution (in addition to the first barrier meant to enforce saturation [10]), whose role looks indeed very similar to that of our above cutoff on the value of T. What was different, however, was the understanding of the physical role played by this barrier: in Ref. [22], the second barrier was merely intended to impose unitarity on those evolution paths which involve a single intermediate point. But it has not been realized there that, for the evolutions leading to a final amplitude $T_f \sim 1$ (as relevant for a study of saturation), the constraint implied by the second barrier is actually sufficient to guarantee unitarity for *arbitrary* paths (i.e. for paths with an arbitrary number of intermediate points). That is, the generality of Eq. (12) as being the correct result in the limit where $Y \to \infty$ and $\alpha_s \to 0$ has not been recognized, nor argued, in Ref. [22].

4   The physical amplitude and its scaling

So far, we have followed the evolution of the amplitude $T(r)$ corresponding to one given partonic realization $|\omega\rangle$. Each such realization undergoes a stochastic evolution given by Eq. (6). At each rapidity, $T(r)$ has the universal shape (8), up to fluctuations concentrated in its tail $T \sim \alpha_s^2$. The position $\rho_s(Y)$ of the front exhibits the Y–dependence [5] given by Eq. (12). However, since the actual evolution is stochastic, the saturation scale ρ_s undergoes a random walk about its mean $\langle\rho_s\rangle$, and ρ_s gets a variance σ. Physically, the origin of this phenomenon can be traced to the statistical fluctuations in the tail $T \sim \alpha_s^2$ of the amplitude: in the course of the evolution, an unusually large number of dipoles may be created, which, after further rapidity evolution, would "pull" the whole front ahead of its typical evolution, resulting in a saturation scale ρ_s for this particular realization larger than the mean $\langle\rho_s\rangle$. This mechanism leads to a spread of the saturation scales of different partonic realizations, while the shape of T in its bulk remains identical. In particular, the velocity of the average front $d\langle\rho_s\rangle/dY$ is still given by Eq. (12) (at least, for sufficiently large Y; see below), because this is the (asymptotic) velocity of all the individual fronts making up the statistical ensemble. This is illustrated in Fig. 2, which in fact applies to the discrete statistical model in Ref. [14] (but a similar situation is expected in QCD): the profile of the amplitude for partonic realizations obtained from different stochastic evolutions over the same rapidity interval are shown. The dispersion of ρ_s is manifest, as well as the universality of the shape of T.

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[5] It is interesting to note in this context that, for a given front realization, the asymptotic velocity, Eq. (12), is reached *exponentially* fast in Y [17], at variance with the mean field problem, where the corresponding approach is only power–like, cf. Eq. (10).

The physical amplitude $A(r, Y)$ is obtained by averaging $T(r)$ over all Fock states at rapidity Y. In order to perform this average, the expression for the variance σ of the saturation scale is needed.

The latter was recently studied in the statistical physics context for various reaction-diffusion like models involving a finite number N of particles. The variance of the position of the front was seen to scale like[6]

$$\sigma^2 = \langle \rho_s^2 \rangle - \langle \rho_s \rangle^2 \sim \frac{\bar{\alpha}Y}{\ln^3(1/\alpha_s^2)} \qquad (13)$$

from numerical simulations [14]. Although to our knowledge there is still no general analytical proof of this result and the status of (13) is still that of a conjecture, such behavior has been checked in independent numerical work (see e.g. Ref. [26]) and is also likely to be very general [17]. We also see on Eq. (13) that the fluctuations of ρ_s, that are of order $\sigma \propto \sqrt{\bar{\alpha}Y}$, are indeed subleading in Y with respect to the effects of discreteness discussed in Sec. 3 (of order $\bar{\alpha}Y$). This is a consistency check of the mean field calculation used to obtain $\langle \rho_s \rangle$, see Eq. (12) .

Knowing the shape (8) of T, the value of the saturation scale ρ_s and the amplitude of its fluctuations σ, we are in a position to evaluate the physical scattering amplitude $A(\rho, Y)$. Up to higher moments of the distribution of ρ_s, $A(\rho, Y)$ is obtained from the amplitudes $T(\rho)|_Y$ for each particular realization of the Fock state of the target at rapidity Y (note that $T(\rho)|_Y$ is implicitly a function of ρ_s, as manifest e.g. in Eq. (8)), after averaging over the corresponding saturation momenta with a Gaussian weight of variance σ (see also Fig. 2):

$$A(\rho, Y) = \frac{1}{\sigma\sqrt{2\pi}} \int d\rho_s \, T(\rho)|_Y \, \exp\left(-\frac{(\rho_s - \langle \rho_s \rangle)^2}{2\sigma^2}\right) \ . \qquad (14)$$

We deduce the following scaling form for the physical amplitude:

$$A(\rho, Y) = A\left(\frac{\rho - \langle \rho_s(Y) \rangle}{\sqrt{\bar{\alpha}Y/\ln^3(1/\alpha_s^2)}}\right) \ , \qquad (15)$$

up to $\mathcal{O}(1/\sqrt{Y})$ corrections. It is obvious from that formula that, at sufficiently high energies, geometric scaling does not hold for the physical amplitude. This feature is a direct consequence of the statistical nature of the parton model: the statistical fluctuations of the number of dipoles translate into a

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[6] Formula (13) is borrowed from the third reference of [14], Eq. (4), with the replacement $\sigma^2 \leftrightarrow D_N \times t$. The time t has to be replaced by $\bar{\alpha}Y$ and the number of particles N is the maximum number of dipoles of a given size $1/\alpha_s^2$.
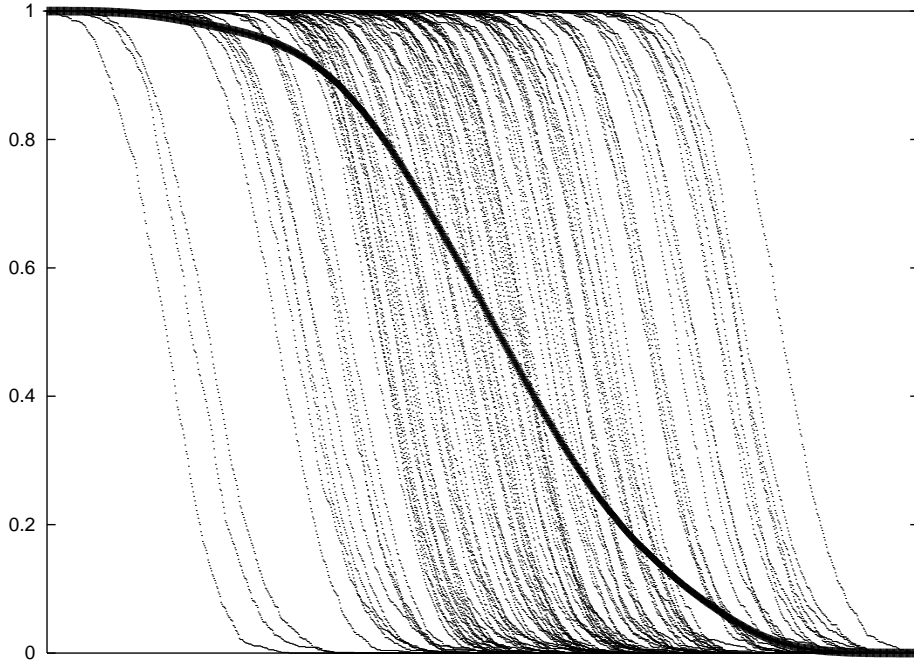
Fig. 2. The scattering amplitude T for different partonic realizations at a given rapidity against $\rho = \ln(r_0^2/r^2)$. The thick line is the average over all realizations, i.e. the physical amplitude A, see Eq. (14).

random wandering of the saturation scale, and after averaging over partonic realizations, the scaling form (15) results.

The violation of geometric scaling manifest in Eq. (15) was already noted in Ref. [22], however, the square root in the denominator of the scaling variable was missing because of a lack of fluctuations in the tail of the distribution: the approach used there was relying on mean field throughout, missing the stochastic nature of the evolution.

5 Conclusion

We have shown that high energy QCD is similar to a reaction-diffusion problem, well studied by statistical physicists. We have been able to obtain the Y-dependence of the saturation scale, see Eq. (12), confirming results obtained recently by different methods [22]. We have also derived the scaling form of the asymptotic dipole-dipole scattering amplitude (15), which is related to the dispersion of saturation scales between different "events" (corresponding to different partonic realizations). That scaling is clearly not geometric.

A recurrent theme in this Letter has been universality that guarantees that the lowest order results do not depend on the details of the model. The exact

way how saturation comes about was not an issue, as well the details of the elementary dipole interaction do not enter the leading order results (in $Y \gg 1$ and $\alpha_s^2 \ll 1$) that we have obtained here. Further terms in these expansions will be model dependent, and thus much more difficult to get.

One of the points that remain to be studied is how fast the computed asymptotics set in. The BK equation may still be a good approximation for a large target (like a nucleus) and in the first stages of the evolution, when the traveling wave front has not diffused down to $T \sim \alpha_s^2$. More precise numerical studies [18,21] may help to clarify this point.

Finally, as mentioned in the Introduction, the dependence upon the impact parameter plays an important role for the overall physical picture of unitarization. This will be discussed at length somewhere else [23].

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